Finite Math - Fall 2018
Lecture Notes - 9/6/2018

Homework

- Section 2.6 - 13, 16, 18, 26, 27, 30, 32, 61, 63, 65, 68

Section 2.6 - Logarithmic Functions

Before we can accurately talk about what logarithms are, let's first remind ourselves about inverse functions.

Inverse Functions. The inverse of a function is given by running the function backwards. But when can we do this?

Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?

Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

This shows that not every function is invertible. To get the inverse of a function, we need each range value to come from *exactly one* domain value. We call such functions *one-to-one*.

If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \xrightarrow{\text{solve for } y} y = f^{-1}(x).$$

Logarithms. We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition 1 (Logarithm)**.** *The* logarithm of base b *is defined as the inverse of* b^x*. That is,*

$$y = b^x \Longleftrightarrow x = \log_b y.$$

Since the domain and range switch when we take inverses, we have

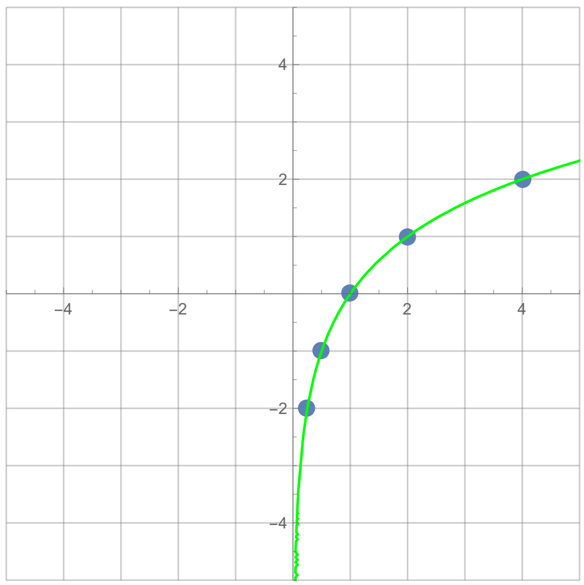
function	domain	range
$f(x) = b^x$	$(-\infty, \infty)$	$(0, \infty)$
$f(x) = \log_b x$	$(0, \infty)$	$(-\infty, \infty)$

Let's look at one example of a graph of a logarithmic function.

Example 1. *Sketch the graph of $f(x) = \log_2 x$.*

Solution. *To get the points for this, we can just recognize that it is the inverse of 2^x so we take each of those points and flip the x and y coordinates. This gives*

x	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$f(x)$	-2	-1	0	1	2



Properties of Logarithms.

Since logarithms are inverse to exponential functions, we get some convenient properties for logarithms:

Property 1 (Properties of Logarithms)**.** *Let $b, M, N > 0$, $b \neq 1$, and p, x be real numbers. Then*

(1) $\log_b 1 = 0$

(2) $\log_b b = 1$

(3) $\log_b b^x = x$

(4) $b^{\log_b x} = x$

(5) $\log_b MN = \log_b M + \log_b N$

(6) $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

(7) $\log_b M^p = p \log_b M$

(8) $\log_b M = \log_b N$ *if and only if* $M = N$

Properties 3 and 7 above are incredibly important to us as we will use them frequently in the study of financial mathematics! Learn these properties well!!

The Natural Logarithm. Just as with exponential functions, if we choose our base to be the number e, we get a special logarithm, the *natural logarithm.*

$$\log_e x = \ln x.$$

We can actually rewrite a logarithm in any base in terms of ln:

$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Using Properties of Logarithms and Exponents.

Example 2. *Solve for x in the following equations:*

(a) $7 = 2e^{0.2x}$

(b) $16 = 5^{3x}$

(c) $8000 = (x-4)^3$

Solution.

(a) *Begin by dividing both sides by 2 to get*

$$3.5 = e^{0.2x}.$$

Next, apply ln *to both sides to get*

$$\ln 7 = \ln e^{0.2x} = 0.2x$$

so that dividing by 0.2 gives

$$x = \frac{\ln 3.5}{0.2} \approx 6.26781.$$

(b) *Straight away, we apply* ln *here to get*

$$\ln 16 = \ln 5^{3x} = 3x \ln 5.$$

Solving for x here gives us

$$x = \frac{\ln 16}{3 \ln 5} \approx 0.57424.$$

(c) *In this example, we have to begin by taking the cube root of both sides, that is, raising both sides to the $\frac{1}{3}$ power:*

$$8000^{\frac{1}{3}} = \left((x-4)^3\right)^{\frac{1}{3}} = x - 4.$$

Then, solving for x gives:

$$x = 8000^{\frac{1}{3}} + 4 = 20 + 4 = 24.$$

A quick reminder of different types of exponents:

- $a^{-n} = \frac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{1/2} = \sqrt{a}$
  - $a^{1/3} = \sqrt[3]{a}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Example 3. *Solve for x in the following equations:*

(a) $75 = 25e^{-x}$

(b) $42 = 7^{2x+3}$

(c) $200 = (2x-1)^5$

Solution.

(a) $x \approx -1.09861$

(b) $x \approx -0.53961$

(c) $x \approx 1.94270$

Applications.

Recall that exponential growth/decay models are of the form

$$A = ce^{rt}.$$

Using the natural logarithm, we can solve for the rate of growth/decay, r, and the time elapsed, t. Let's see this in an example.

Example 4. *The isotope carbon-14 has a half-life (the time it takes for the isotope to decay to half of its original mass) of 5730 years.*

(a) *At what rate does carbon-14 decay?*

(b) *How long would it take for 90% of a chunk of carbon-14 to decay?*

Solution.

(a) *Suppose we have an initial mass of M_0. After half of it decays, the mass will be $\dfrac{M_0}{2}$ and this happens after $t = 5730$ years has elapsed. Plugging all this into our model, we get*

$$\frac{M_0}{2} = M_0 e^{r(5730)} \iff \frac{1}{2} = e^{5730r}$$

Applying the natural log to each side gives

$$\ln \frac{1}{2} = \ln e^{5730r}$$

Using properties of logarithms, we have

$$\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$$

and

$$\ln e^{5730r} = 5730r \ln e = 5730r$$

so that

$$-\ln 2 = 5730r.$$

Solving for r*, we get*

$$r = -\frac{\ln 2}{5730} \approx -0.00012$$

*This means that carbon-*14 *decays at a rate of* 0.12% *per year.*

(b) *If the mass of* M_0 *loses* 90% *of its mass, we're looking for the time it takes for only* $0.1M_0$ *to remain. So,*

$$0.1M_0 = M_0 e^{-0.00012t}$$

and canceling the M_0*'s gives*

$$0.1 = e^{-0.00012t}.$$

Hit both sides of this with ln *to get*

$$\ln 0.1 = \ln e^{-0.00012t} = -0.00012t.$$

Solve for t

$$t = -\frac{\ln 0.1}{0.00012} \approx 19,188.21.$$

So, it would take about 19,188.21 *years for* 90% *of the original mass to decay.*